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ANALYSIS OF A PARTICULAR STOCHASTIC DIFFERENTIAL EQUATION

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The stochastic differential equation and its analogous difference equation for a damped, oscillatory system subject to random impulses are discussed. For a specific set of conditions on the coefficients of the differential equation, the autocorrelation function and the relations among the coefficients of the differential and difference equations are obtained.

INTRODUCTION

Bartlett [2] (*pp. 144–52, 265–69*) discusses in some detail the stochastic differential equation

$$d\dot{X}(t) + \alpha\dot{X}(t)dt + \beta X(t)dt = dY(t), \tag{1}$$

and its analogous difference equation

$$X_{t+2} + aX_{t+1} + bX_t = Y_{t+2}. \tag{2}$$

Equation (1) may be considered as representing a damped, oscillatory system which is subjected to random impulses. The notation $dY(t)$ represents the differential, in the mean square sense, of the additive process

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The Pioneering Research Laboratory on Physics of Fine Particles was established in 1962, at the Ohio Agricultural Experiment Station, Wooster, by the Agricultural Research Service of the United States Department of Agriculture, in recognition of the need for continued fundamental research on fine particle behavior and, in particular, the criticality of such studies to agriculture.

In agriculture, pesticides in both liquid droplet and solid particle form are used extensively, but their use is frequently hampered by lack of optimum efficiency and precision of application, despite the best efforts of manufacturers and applicators. This results in an excessive economic burden on agriculture in the maintenance of the best of quality in agricultural produce, as well as an increased chance of contaminating surrounding areas. Problems of air pollution are also of prime pertinence and concern. It is desired that the results of these investigations in fine particle behavior and its inherent subject matter will serve agriculture and its allied industries, and other industries with similar vexations, in the alleviation of these problems.

[2] M. S. Bartlett, *An Introduction to Stochastic Processes*. 312 pp. London. 1956.

$Y(t)$ of the accumulated impulse effects. The impulses act to change the velocity $\dot{X}(t)$ discontinuously. Bartlett gives the solutions for these equations as

$$X(t)=A(u)e^{\lambda_1(t-u)}+B(u)e^{\lambda_2(t-u)}+\int_u^t \frac{e^{\lambda_1(t-v)}-e^{\lambda_2(t-v)}}{\lambda_1-\lambda_2}\,dY(v) \tag{3}$$

for (1) and

$$X_r=A\mu_1{}^{r-u}+B\mu_2{}^{r-u}+\sum_{s=0}^{r-u-2}\frac{\mu_1{}^{s+1}-\mu_2{}^{s+1}}{\mu_1-\mu_2}Y_{r-s} \tag{4}$$

for (2) Here, $A(u)$ and $B(u)$ are determined from the values of $X(t)$, $\dot{X}(t)$ at $t=u$, and A and B are obtained from the values X_u and X_{u+1} at u, $u+1$. The values λ_1 and λ_2 are roots of the equation

$$\lambda^2+\alpha\lambda+\beta=O, \tag{5}$$

and μ_1 and μ_2 are roots of the equation

$$E_r^2+aE_r+b=O, \tag{6}$$

where $E_r=1+\Delta$ is the displacement operator in the calculus of finite differences. If $|e^{\lambda_1}|<1$ and $|e^{\lambda_2}|<1$, $X(t)$ becomes independent of $X(u)$ as $u\rightarrow-\infty$ and (3) becomes

$$X(t)=\int_{-\infty}^t \frac{e^{\lambda_1(t-u)}-e^{\lambda_2(t-u)}}{\lambda_1-\lambda_2}\,dY(u). \tag{7}$$

Similarly, if $|\mu_1|<1$ and $|\mu_2|<1$, then as $u\rightarrow-\infty$. (4) becomes

$$X_r=\sum_{s=0}^{r-u-2}\frac{\mu_1{}^{s+1}-\mu_2{}^{s+1}}{\mu_1-\mu_2}Y_{r-s}. \tag{8}$$

The above system exhibits a damped correlogram when subjected to a time-series analysis. The coefficients a and b may be estimated by least squares methods, and if the coefficients α and β are estimated from them, then the theoretical correlogram may be more conveniently calculated. Bartlett gives the appropriate procedure for the case $\beta\geqslant\frac{1}{4}\alpha^2$. In the remainder of this paper, we shall discuss the case $\beta\leqslant\frac{1}{4}\alpha^2$.

THEORY

Bartlett[3] indicates that the covariance function $w(\tau)$ for (3) is given by

$$w(\tau)=\sigma_x{}^2\left(\frac{\lambda_1e^{\lambda_2\tau}-\lambda_2e^{\lambda_1\tau}}{\lambda_1-\lambda_2}\right), \tag{9}$$

where σ_x^2 is the variance

$$\sigma_x{}^2=E\{X^*(t)X(t)\}, \tag{10}$$

with $X^*(t)$ representing the complex conjugate of the process $X(t)$. Since

$$w(\tau)=\sigma_x^2\,\rho(\tau), \tag{11}$$

[3] See page 149 of footnote 2.

where $\rho(\tau)$ is the autocorrelation function of $X(t)$, then

$$\rho(\tau)=\frac{\lambda_1 e^{\lambda_2 \tau}-\lambda_2 e^{\lambda_1 \tau}}{\lambda_1-\lambda_2}. \tag{12}$$

For the case $\alpha^2 \geqslant 4\beta$, we obtain from (5) that

$$\lambda_1=-\gamma+(\gamma^2-\beta)^{1/2} \tag{13}$$

and

$$\lambda_2=-\gamma-(\gamma^2-\beta)^{1/2}, \tag{14}$$

where γ is defined as

$$\gamma=\alpha/2. \tag{15}$$

We also define

$$\lambda_0=(\gamma^2-\beta)^{1/2},\ (\lambda_0 \geqslant 0). \tag{16}$$

Hence, upon insertion of (13) and (14) into (12), we find after some rearrangement that

$$\rho(\tau)=e^{-\gamma\tau}\left\{\frac{\gamma}{\lambda_0}\left[\frac{e^{\lambda_0\tau}-e^{-\lambda_0\tau}}{2}\right]+\left[\frac{e^{\lambda_0\tau}+e^{-\lambda_0\tau}}{2}\right]\right\}, \tag{17}$$

which is equivalent to

$$\rho(\tau)=e^{-\gamma|\tau|}\left(\frac{\gamma}{\lambda_0}\sinh \lambda_0|\tau|+\cosh \lambda_0|\tau|\right) \tag{18}$$

for a stationary process.

We now write the definition

$$\rho(\tau)=Ae^{-\gamma|\tau|}\cosh(\lambda_0|\tau|-\theta). \tag{19}$$

Since we require that $\rho(0)=1$, then it follows that

$$A=\frac{1}{\cosh\theta}, \tag{20}$$

and

$$\rho(\tau)=\frac{e^{-\gamma|\tau|}\cosh(\lambda_0|\tau|-\theta)}{\cosh\theta}. \tag{21}$$

From (21) and (18), it is possible to write

$$\rho(\tau)=e^{-\gamma|\tau|}\left\{\frac{\cosh\lambda_0|\tau|\cosh\theta}{\cosh\theta}+\frac{\sinh\lambda_0|\tau|\sinh\theta}{\cosh\theta}\right\}=e^{-\gamma|\tau|}(\cosh\lambda_0|\tau|+\tanh\theta\sinh\lambda_0|\tau|)$$

$$=e^{-\lambda|\tau|}(\cosh\lambda_0|r|+\frac{\gamma}{\lambda_0}\sinh\lambda_0|\tau|),$$

which gives

$$\theta=\tanh^{-1}\frac{\gamma}{\lambda_0}. \tag{22}$$

Eqs. (21), (22), (15), and (16) afford a means of computing the theoretical correlogram. We now determine the relationships among the coefficients of (1) and (2) for the case under consideration, viz., $\alpha^2 \geqslant 4\beta$.

The autocorrelation function $\rho(\tau)$ satisfies the difference equation

$$\rho(\tau+2h)+a\rho(\tau+h)+b\rho(\tau)=O,\ (\tau \geqslant O). \tag{23}$$

Since $\rho(O)=1$, we have from (23) the requirement

$$\rho(2h)+a\rho(h)+b=O. \tag{24}$$

If (21) is inserted in (24), we find after some manipulation that

$$e^{-\alpha h} \cosh (2\lambda_0 h) \cosh \theta - e^{-\alpha h} \sinh (2\lambda_0 h) \sinh \theta$$
$$= -[ae^{-\alpha h/2} \cosh (\lambda_0 h) - b \cosh \theta + ae^{-\alpha h/2} \sinh (\lambda_0 h) \sinh \theta. \tag{25}$$

Equating the coefficients of cosh θ and sinh θ, we obtain the pair of equations

$$ae^{-\alpha h/2} \cosh (\lambda_0 h) + b = - e^{-\alpha h} \cosh (2\lambda_0 h) \tag{26}$$

and

$$ae^{-\alpha h/2} \sinh (\lambda_0 h) = - e^{-\alpha h} \sinh (2\lambda_0 h). \tag{27}$$

These equations are easily solved to yield

$$a = - 2e^{-\alpha h/2} \cosh (\lambda_0 h) \tag{28}$$

and

$$b = e^{-\alpha h}. \tag{29}$$

Eqs. (28) and (29) with (16) permit estimation of α and β from the least squares estimates of a and b.

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